COLORINGS OF SIMPLICIAL COMPLEXES AND VERTEX DECOMPOSABILITY

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Abstract. In attempting to understand how combinatorial modifications alter algebraic properties of monomial ideals, several authors have investigated the process of adding "whiskers" to graphs. The first and the fourth authors developed a similar construction to build a vertex decomposable simplicial complex Δ_χ from a coloring χ of the vertices of a simplicial complex Δ. In this paper, we study this construction for colorings of subsets of the vertices, and give necessary and sufficient conditions for this construction to produce vertex decomposable simplicial complexes. Using combinatorial topology, we strengthen and give new proofs for results of the second and third authors about sequentially Cohen-Macaulay edge ideals that were originally proved using algebraic techniques.

1. Introduction

Square-free monomial ideals are intimately connected to combinatorics. This connection raises the natural question: how do changes in combinatorial structures affect algebraic properties of associated square-free monomial ideals? In [15], Villarreal investigates the process of adding whiskers to a finite simple graph, and (citing also Fröberg, Herzog, and Vasconcelos) proves that the edge ideal of a graph with whiskers added to every vertex is always Cohen-Macaulay. To add a *whisker* to a vertex, one adds an additional vertex and an edge between the old vertex and the new one.

Generalizing Villarreal's work, in [8] the second and the third authors studied additions of whiskers to subsets of the vertices that produce sequentially Cohen-Macaulay edge ideals. The configuration of the whiskers, not the number, determines when the resulting ideals are sequentially Cohen-Macaulay, demonstrating the subtlety of the problem. The techniques in [8] are mostly algebraic, focusing on when the cover ideals are componentwise linear, a property Alexander dual to sequentially Cohen-Macaulayness.

In a different direction, several authors have used methods from combinatorial topology to study similar phenomena. The primary combinatorial object in these efforts is the independence complex of a graph, the simplicial complex whose Stanley-Reisner ideal

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equals the edge ideal of the graph. For instance, Woodroofe [16] and Dochtermann and Engström [7] use combinatorial topology to prove that the independence complex of a chordal graph is vertex decomposable, implying that the edge ideal is sequentially Cohen-Macaulay. Dochtermann and Engström [7] also show that the independence complex of a whiskered graph is a pure vertex decomposable simplicial complex, and consequently, Cohen-Macaulay, thus giving a combinatorial topological proof of Villarreal's result. Cook and Nagel [6] use full clique-whiskering, a technique that begins by partitioning the vertex set of a graph into cliques. For each of these cliques, one adds a new vertex and connects it to each vertex in the clique. Cook and Nagel prove that fully clique-whiskered graphs are vertex-decomposable [6, Theorem 3.3]; when the cliques in the partition each consist of a single vertex, this recovers the results of Villarreal and Dochtermann and Engström.

The first and fourth authors take a blended approach in [1] to extend these results about independence complexes of graphs to all simplicial complexes. Starting with any coloring χ of the vertices of Δ, they construct a new simplicial complex Δ_χ that is vertex decomposable. The whiskering construction of Villarreal and the clique-whiskering technique of Cook and Nagel [6] become special instances of this construction.

The constructions in [1, 6, 15] always result in *pure* vertex decomposable simplicial complexes (and thus, Cohen-Macaulay complexes). The algebraic results of [8] are therefore not a consequence of these results because the corresponding independence complex associated to the partially whiskered graph is not necessarily pure.

In this paper, in the spirit of [8], we extend the construction in [1] to partial whiskerings of simplicial complexes. We start with a partial coloring χ of Δ (see Definition 2.9) and use χ and Δ to build a new simplicial complex Δ_χ (see Construction 3.1), which we call a partially whiskered simplicial complex. The main result of this paper is a necessary and sufficient condition for a partially whiskered simplicial complex to be vertex decomposable.

Theorem 1.1 (Theorem 3.3)**.** *Let Δ be a simplicial complex on the vertex set V and let W be a subset of V. Let χ be the s-coloring of $\Delta|_W$ given by $W = W_1 \cup \cdots \cup W_s$. Then Δ_χ is vertex decomposable if and only if* $\mathrm{link}_\Delta(\mu)|_{\overline{W}}$ *is vertex decomposable for every face μ of Δ such that $\mu \subseteq W$.*

Theorem 3.3 has a number of consequences. Corollary 3.6 shows that when $W = V$, Δ_χ is always vertex decomposable, thus recovering the main result of [1]. Corollary 3.6 also gives the analog to the numerical bound of the second and third authors [8] for graphs. Namely, if one has a simplicial complex with n vertices, and $|W| \geq n - 3$, one gets a vertex decomposable simplicial complex. As in the case of graphs, this bound is sharp (see Example 3.7).

In Section 4, we specialize Theorem 3.3 to the case of independence complexes. In particular, we get necessary and sufficient conditions for a whiskered graph to be vertex decomposable (see Theorem 4.4). This result yields Corollary 4.6, a combinatorial proof for [8, Theorem 3.3]; this provides the combinatorial approach to the results of [8] sought in [7].

Our paper is organized as follows. In Section 2, we recall the relevant background. In Section 3 we present the main theorems and derive some of their consequences. Section 4 applies our results to independence complexes of graphs.

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2. Background

We recall the relevant background on simplicial complexes.

Definition 2.1. A finite *simplicial complex* Δ on a finite vertex set V is a collection of subsets of V with the property that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. The elements of Δ are called *faces*. The maximal faces of Δ, with respect to inclusion, are the *facets*.

The vertex sets of our simplicial complexes will be either the set $\{x_1, \ldots, x_n\}$ or the set $\{x_1, \ldots, x_n, y_1, \ldots, y_s\}$. If Δ is a simplicial complex and $\sigma \in \Delta$, then we say σ has *dimension* d if $|\sigma| = d + 1$ (by convention, the empty set has dimension -1). We say Δ is *pure* if all of its facets have the same dimension, otherwise Δ is *non-pure*. If $F_1, \ldots, F_t$ is a complete list of the facets of Δ, we sometimes write Δ as $\Delta = \langle F_1, \ldots, F_t\rangle$. If $\sigma \in \Delta$ is a face, then the *deletion* of σ from Δ is the simplicial complex defined by

$$\Delta \setminus \sigma = \{\tau \in \Delta \mid \sigma \not\subseteq \tau\}.$$

The *link* of σ in Δ is the simplicial complex defined by

$$\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\}.$$

When $\sigma = \{v\}$, we shall abuse notation and write $\Delta \setminus v$ (respectively $\mathrm{link}_\Delta(v)$) for $\Delta \setminus \{v\}$ (respectively $\mathrm{link}_\Delta(\{v\})$).

We shall be particularly interested in the class of vertex decomposable simplicial complexes. This class was first introduced in the pure case by Provan and Billera [13] and in the non-pure case by Björner and Wachs [3]. We recall the non-pure version.

Definition 2.2. A simplicial complex Δ on vertex set V is called *vertex decomposable* if

(1) Δ is a simplex, or
(2) there is some vertex $v \in V$ such that $\Delta \setminus v$ and $\mathrm{link}_\Delta(v)$ are vertex decomposable and do not share any facets; such a vertex v is called a *shedding* vertex of Δ.

Remark 2.3. When Δ is vertex decomposable, then Δ also inherits other combinatorial and algebraic properties. In particular, if Δ is pure and vertex decomposable, then Δ has a pure shelling, and its Stanley-Reisner ring R/I_Δ is Cohen-Macaulay. If Δ is non-pure and vertex decomposable, then Δ is still shellable, in the non-pure sense of Björner and Wachs [3], and its Stanley-Reisner ring R/I_Δ is sequentially Cohen-Macaulay. We point the reader to the text of Herzog and Hibi [11] for a complete treatment of these ideas.

For simplicial complexes Δ and Ω over disjoint vertex sets V and U, respectively, the *join* of Δ and Ω, denoted by $\Delta \cdot \Omega$, is the simplicial complex over the vertex set $V \cup U$, whose faces are $\{\sigma \cup \tau \mid \sigma \in \Delta, \tau \in \Omega\}$. Provan and Billera proved:

Theorem 2.4 ([13, Proposition 2.4])**.** *The join $\Delta \cdot \Omega$ is vertex decomposable if and only if both Δ and Ω are vertex decomposable.*

The property of vertex decomposability is preserved when forming a link. The following result was first proved in [13, Proposition 2.3] in the pure case; the non-pure case follows similarly, as noted in [12, Theorem 3.30] and [17, Proposition 3.7].

Theorem 2.5. *If Δ is vertex decomposable, then $\operatorname{link}_\Delta(\sigma)$ is vertex decomposable for any $\sigma \in \Delta$.*

An important notion for our main construction and results in Section 3 is that of a coloring of a simplicial complex.

Definition 2.6. Let Δ be a simplicial complex on the vertex set V with facets $F_1, \ldots, F_t$. An *s-coloring* of Δ is a partition of the vertices $V = V_1 \cup \cdots \cup V_s$ (where the sets V_i are allowed to be empty) such that $|F_i \cap V_j| \leq 1$ for all $1 \leq i \leq t, 1 \leq j \leq s$. We will sometimes write that χ is an s-coloring of Δ to mean χ is a specific partition of V that gives an s-coloring of Δ. If there exists an s-coloring of Δ, we say that Δ is *s-colorable.*

Example 2.7. If Δ is a simplicial complex on $|V| = n$ vertices, then Δ is n-colorable; indeed, we take our coloring to be $V = \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\}$.

In this paper, we are interested in the case in which a subset of the vertices of a simplicial complex is colored, or equivalently, in the coloring of an induced subcomplex.

Definition 2.8. Let Δ be a simplicial complex on the vertex set V and let $W \subseteq V$. The *restriction of* Δ *to* W (or equivalently, the *induced subcomplex over* W) is the subcomplex

$$\Delta|_W = \{\sigma \in \Delta \mid \sigma \subseteq W\}.$$

Definition 2.9. Let Δ be a simplicial complex on the vertex set V and let $W \subseteq V$. If χ is an s-coloring of the restriction $\Delta|_W$, then we call χ a *partial coloring* of Δ. We call the vertices in W the *colored vertices* of Δ and those in $\overline{W} = V \setminus W$ the *non-colored vertices.*

3. Partial colorings and vertex decomposability

Given any simplicial complex Δ with a partial coloring χ, we introduce a construction to make a new simplicial complex Δ_χ. Our main result gives necessary and sufficient conditions for Δ_χ to be vertex decomposable. We begin with the construction of Δ_χ.

Construction 3.1. Let Δ be a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$, and let W be a subset of V. Let χ be an s-coloring of $\Delta|_W$ given by $W = W_1 \cup \cdots \cup W_s$. Define Δ_χ to be the simplicial complex on the vertex set $\{x_1, \ldots, x_n, y_1, \ldots, y_s\}$ with faces

$\sigma \cup \tau$, where σ is a face of Δ, and τ is any subset of $\{y_1, \ldots, y_s\}$ such that for all $y_j \in \tau$, we have $\sigma \cap W_j = \emptyset$.

The first and fourth authors recently studied Construction 3.1 in [1] in the case that $V = W$; this case also appears in [9] and implicitly in a proof in [2]. We call the process of adding a new vertex y_i for a color class W_i *whiskering*, and call the resulting complex Δ_χ the *(partially) whiskered* simplicial complex.

Remark 3.2. To forestall any potential confusion, "whiskering a vertex," as first defined in [15], referred to adding a new additional vertex to a graph and joining the new and old vertices by an edge. This operation was defined in terms of the finite simple graph. However, this procedure also results in a change in the independence complex of the graph (see Section 4 for details). In our definition, when we use the term "whiskering", we are generalizing the operation that changes the independence complex, not the graph.

Our main result is necessary and sufficient conditions for Δ_χ to be vertex decomposable.

Theorem 3.3. *Let Δ be a simplicial complex on the vertex set V, and let W be a subset of V. Let χ be the s-coloring of $\Delta|_W$ given by $W = W_1 \cup \cdots \cup W_s$. Then Δ_χ is vertex decomposable if and only if* $\mathrm{link}_\Delta(\mu)|_{\overline{W}}$ *is vertex decomposable for every face μ of Δ such that $\mu \subseteq W$.*

Remark 3.4. The special case $\mu = \emptyset$ in Theorem 3.3 is instructive. Because $\mathrm{link}_\Delta(\emptyset) = \Delta$, the link hypothesis imposes the condition that $\Delta|_{\overline{W}}$ is vertex decomposable.

Proof of Theorem 3.3. ($\Leftarrow$) We proceed by induction on the number of vertices of Δ. The base case is the empty simplicial complex $\Delta = \{\emptyset\}$. In this case the only partial coloring of Δ is $W = W_1 \cup \cdots \cup W_s$, where all the W_i are empty. Then Δ_χ is the simplex $\langle \{y_1, \ldots, y_s\} \rangle$, which is vertex decomposable.

Now let Δ be a simplicial complex on vertex set $V = \{x_1, \ldots, x_n\}$, and let $W \subseteq V$. If $W = \emptyset$, then $\Delta_\chi = \Delta \cdot \langle \{y_1, \ldots, y_s\} \rangle$, the join of two simplicial complexes. The link hypothesis with $\mu = \emptyset$ implies that $\Delta|_{\overline{\emptyset}} = \Delta$ is vertex decomposable. It then follows from Theorem 2.4 that Δ_χ is vertex decomposable. We can therefore assume that $\Delta \neq \{\emptyset\}$ and $W \neq \emptyset$.

Let $w \in W$. After relabelling, we will assume that $w \in W_1$. To prove that Δ_χ is vertex decomposable, we will show that $\Delta_\chi \setminus w$ and $\mathrm{link}_{\Delta_\chi}(w)$ are both vertex decomposable.

Recall that the faces of Δ_χ are of the form $\sigma \cup \tau$, where σ is a face of Δ and $\tau \subseteq \{y_1, \ldots y_s\}$ such that if $y_j \in \tau$, then $W_j \cap \sigma = \emptyset$. Note that

$$\begin{aligned} \Delta_\chi \setminus w &= \{\sigma \cup \tau \in \Delta_\chi \mid w \notin \sigma \cup \tau\} \\ &= \{\sigma \cup \tau \in \Delta_\chi \mid w \notin \sigma\} \\ &= (\Delta \setminus w)_{\chi'} \end{aligned}$$

where χ' is the partial coloring $(W_1\setminus\{w\})\cup W_2\cup\cdots\cup W_s$ of $\Delta\setminus w$ induced by the coloring χ of Δ. Since $w\in W$, the uncolored vertices of Δ and $\Delta\setminus w$ are the same set $\overline{W}$. Now let μ be a face of $\Delta\setminus w$ such that $\mu\subseteq (W\setminus\{w\})$. Then since $w\notin\overline{W}$,

$$\begin{aligned}(\mathrm{link}_{\Delta\setminus w}(\mu))|_{\overline{W}} &= \{\sigma\in(\Delta\setminus w)\mid \mu\cap\sigma=\emptyset, \mu\cup\sigma\in(\Delta\setminus w)\}|_{\overline{W}}\\ &= \{\sigma\in\Delta\mid w\notin\sigma, \mu\cap\sigma=\emptyset, \mu\cup\sigma\in(\Delta\setminus w)\}|_{\overline{W}}\\ &= \{\sigma\in\Delta\mid \mu\cap\sigma=\emptyset, \mu\cup\sigma\in\Delta\}|_{\overline{W}}\\ &= (\mathrm{link}_{\Delta}(\mu))|_{\overline{W}}.\end{aligned}$$

Thus we have that $(\mathrm{link}_{\Delta\setminus w}(\mu))|_{\overline{W}}$ is vertex decomposable by hypothesis. Therefore, since $\Delta\setminus w$ is a simplicial complex on fewer than n vertices with χ' a partial coloring on $W\setminus\{w\}$ such that $(\mathrm{link}_{\Delta\setminus w}(\mu))|_{\overline{W}}$ is vertex decomposable for all $\mu\subseteq(W\setminus\{w\})$, induction implies that $\Delta_\chi\setminus w=(\Delta\setminus w)_{\chi'}$ is vertex decomposable.

Now consider $\mathrm{link}_{\Delta_\chi}(w)$. We have

$$\begin{aligned}\mathrm{link}_{\Delta_\chi}(w) &= \{\sigma\cup\tau\in\Delta_\chi\mid w\notin\sigma\cup\tau, \sigma\cup\tau\cup\{w\}\in\Delta_\chi\}\\ &= \{\sigma\cup\tau\in\Delta_\chi\mid w\notin\sigma, (\sigma\cup\{w\})\cup\tau\in\Delta_\chi\}\\ &= (\mathrm{link}_\Delta(w))_{\chi''}\end{aligned}$$

where χ'' is the partial coloring of $\mathrm{link}_\Delta(w)$ given by $U_2\cup\cdots\cup U_s$ where $U_j=W_j\cap\{\text{vertices of } \mathrm{link}_\Delta(w)\}$. Because $w\in W_1$, we need only consider U_j for $j=2,\ldots,s$. Set $U=U_2\cup\cdots\cup U_s$ (i.e., the colored vertices of $\mathrm{link}_\Delta(w)$) and $\overline{U}$ to be the set of non-colored vertices of $\mathrm{link}_\Delta(w)$.

Note that $\overline{U}=\overline{W}\cap\{\text{vertices of } \mathrm{link}_\Delta(w)\}$. Let μ be a face of $\mathrm{link}_\Delta(w)$ such that $\mu\subseteq U$. Since $\mu\in\mathrm{link}_\Delta(w)$, we have $\mu\cup\{w\}\in\Delta$. Then

$$\mathrm{link}_{\mathrm{link}_\Delta(w)}(\mu)=\mathrm{link}_\Delta(\mu\cup\{w\})$$

so

$$(\mathrm{link}_{\mathrm{link}_\Delta(w)}(\mu))|_{\overline{U}}=(\mathrm{link}_{\mathrm{link}_\Delta(w)}(\mu))|_{\overline{W}}=(\mathrm{link}_\Delta(\mu\cup\{w\}))|_{\overline{W}}.$$

By the assumption on Δ, $(\mathrm{link}_\Delta(\mu\cup\{w\}))|_{\overline{W}}$ is vertex decomposable. Because $\mathrm{link}_{\Delta_\chi}(w)=(\mathrm{link}_\Delta(w))_{\chi''}$ and $\mathrm{link}_\Delta(w)$ is a simplicial complex on fewer than n vertices, $\mathrm{link}_{\Delta_\chi}(w)$ is vertex decomposable by induction.

To show that Δ_χ is vertex decomposable, all that remains is to show that no facet of $\mathrm{link}_{\Delta_\chi}(w)$ is a facet of $\Delta_\chi\setminus w$. Let $\sigma\cup\tau$ be a facet of $\mathrm{link}_{\Delta_\chi}(w)$. Then $(\sigma\cup\{w\})\cup\tau$ is a face of Δ_χ, so $y_1\notin\tau$. On the other hand, if $\sigma\cup\tau\in\Delta_\chi\setminus w$, then since $\sigma\cap W_1=\emptyset$, $\sigma\cup\tau\cup\{y_1\}$ is also a face of $\Delta_\chi\setminus w$ and so the link and deletion do not share any facets.

($\Rightarrow$) Let $\mu\in\Delta$ be a face such that $\mu\subseteq W$, and hence $\mu\in\Delta|_W$. Because χ is an s-coloring of $\Delta|_W$, we have $|\mu\cap W_i|\leq 1$ for $i=1,\ldots,s$. After relabelling the W_j's, we may assume that $|\mu\cap W_i|=1$ for $i=1,\ldots,t$, and $|\mu\cap W_i|=0$ for $i=t+1,\ldots,s$.

By Construction 3.1 we have $\mu\cup\{y_{t+1},\ldots,y_s\}\in\Delta_\chi$. We now claim that

$$\mathrm{link}_\Delta(\mu)|_{\overline{W}}=\mathrm{link}_{\Delta_\chi}(\mu\cup\{y_{t+1},\ldots,y_s\}).$$

Notice that our conclusion will then follow from this claim and Theorem 2.5 because Δ_χ is assumed to be vertex decomposable.

For any $\tau \in \operatorname{link}_\Delta(\mu)|_{\overline{W}}$, we have $\tau \cup \mu \in \Delta$, $\tau \cap \mu = \emptyset$, and $\tau \cap W = \emptyset$. By Construction 3.1, $\tau \cup \mu \cup \{y_{t+1}, \ldots, y_s\} \in \Delta_\chi$ because $(\tau \cup \mu) \cap W_i = \emptyset$ for $i = t+1, \ldots, s$. Moreover, since $\tau \cap (\mu \cup \{y_{t+1}, \ldots, y_s\}) = \emptyset$, we have $\tau \in \operatorname{link}_{\Delta_\chi}(\mu \cup \{y_{t+1}, \ldots, y_s\})$.

We now consider the reverse inclusion. Let $\tau \in \operatorname{link}_{\Delta_\chi}(\mu \cup \{y_{t+1}, \ldots, y_s\})$. Thus, $\tau \cup \mu \cup \{y_{t+1}, \ldots, y_s\} \in \Delta_\chi$ and $\tau \cap (\mu \cup \{y_{t+1}, \ldots, y_s\}) = \emptyset$. Since $|\mu \cap W_i| = 1$ for $i = 1, \ldots, t$ we know that $y_i \notin \tau$ for $i = 1, \ldots, t$. Therefore $\tau \subseteq \{x_1, \ldots, x_n\}$, and $\tau \cup \mu$ must be a face of Δ. Thus $\tau \in \operatorname{link}_\Delta(\mu)$.

Further, since $\tau \cup \mu \in \Delta$ and $|\mu \cap W_i| = 1$ for $i = 1, \ldots, t$, we have $|\tau \cap W_i| = 0$ for $i = 1, \ldots, t$. Since $\tau \cup \mu \cup \{y_{t+1}, \ldots, y_s\} \in \Delta_\chi$ we have $|\tau \cap W_i| = 0$ for $i = t+1, \ldots, s$ as well. Therefore $\tau \in \operatorname{link}_\Delta(\mu)|_{\overline{W}}$. $\square$

Remark 3.5. Let $\Delta = \langle x_1x_2x_3x_4, x_1x_3x_4x_5, x_1x_3x_5x_6, x_1x_2x_5x_6, x_2x_3x_6 \rangle$, and let χ be the coloring given by $W = \{x_1\} \cup \{x_2\}$. Then $\Delta|_{\overline{W}}$, $\operatorname{link}_\Delta(x_1)|_{\overline{W}}$, and $\operatorname{link}_\Delta(x_2)|_{\overline{W}}$ are all vertex decomposable. However, $\operatorname{link}_\Delta(\{x_1, x_2\})|_{\overline{W}}$ is not vertex decomposable, so by Theorem 3.3, neither is Δ_χ.

We now give a bound on the number of vertices to color to ensure that Δ_χ is vertex decomposable. The following corollary also recovers [1, Theorem 3.7] in the case where $|V \setminus W| = 0$.

Corollary 3.6. *Let Δ be a simplicial complex on vertex set V, W a subset of V, and χ a coloring of $\Delta|_W$. If $|V \setminus W| \leq 3$, then Δ_χ is vertex decomposable.*

Proof. All simplicial complexes on three or fewer vertices are vertex decomposable. Since $|\overline{W}| = |V \setminus W| \leq 3$, $\operatorname{link}_\Delta(\mu)|_{\overline{W}}$ is vertex decomposable for any $\mu \in \Delta$ such that $\mu \subseteq W$. Thus, by Theorem 3.3, Δ_χ is vertex decomposable. $\square$

The previous corollary is an analog of a bound of the second and third authors [8, Corollary 3.5]. The numerical bound on the cardinality of $\overline{W}$ in Corollary 3.6 is sharp:

Example 3.7. Let $\Delta = \langle x_1x_2x_3, x_3x_4x_5 \rangle$, $W = \{x_3\}$ and χ be the coloring of $\Delta|_W$ given by $W = W_1 = \{x_3\}$, so $|\overline{W}| = 4$. Then $\operatorname{link}_\Delta(\emptyset) = \Delta|_{\overline{W}} = \langle x_1x_2, x_4x_5 \rangle$, which is not vertex decomposable. Thus Δ with this coloring does not fit the conditions of Theorem 3.3. Indeed, $\Delta_\chi = \langle x_4x_5y_1, x_1x_2y_1, x_3x_4x_5, x_1x_2x_3 \rangle$ is not vertex decomposable.

As noted in [8], it is not necessarily the number of whiskers but rather their configuration that determines the sequentially Cohen-Macaulayness of the resulting ideals. A similar phenomenon occurs in our setting. In particular, given any coloring of Δ, if χ is its restriction to all but one color class, then Δ_χ is vertex decomposable.

Corollary 3.8. *Let Δ be a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$ and let χ be an s-coloring of Δ given by $V = V_1 \cup \cdots \cup V_s$. For each $i = 1, \ldots, s$, let χ_i be the*

induced partial coloring of $\Delta|_{Y_i}$ *given by* $Y_i = V_1 \cup \cdots V_{i-1} \cup V_{i+1} \cup \cdots V_s$. *Then* Δ_{χ_i} *is vertex decomposable for each* $i = 1, \ldots, s$.

Proof. It suffices to prove the statement for $i = 1$. Let $Y_1 = V_2 \cup \cdots \cup V_s$ be the induced partial coloring of Δ given by χ_1. Then $\overline{Y_1} = V_1$. Since χ is a coloring, if $\sigma \in \Delta$ and $\sigma \subseteq V_1$, then $|\sigma| \leq 1$. Then for any $\mu \subseteq Y$, $\mathrm{link}_\Delta(\mu)|_{V_1}$ is either the simplicial complex $\{\emptyset\}$ or a zero-dimensional simplex. Because these simplicial complexes are vertex decomposable, Theorem 3.3 implies the desired result. □

4. Independence Complexes

We round out this paper by applying Theorem 3.3 to independence complexes of graphs. In particular, we give a new combinatorial proof of [8, Theorem 3.3].

We recall some terminology. Let $G = (V, E)$ be a finite simple graph. We say $W \subseteq V$ is an *independent set* if for all $e \in E$, $e \cap W \neq e$. We can form a simplicial complex from the independent sets of G:

Definition 4.1. Let G be a finite simple graph. The *independence complex* of G, denoted $\mathrm{Ind}(G)$, is the simplicial complex $\mathrm{Ind}(G) = \{W \mid W \text{ is an independent set of } G\}$.

As described in the introduction, given a subset $S \subseteq V$, we denote by $G \cup W(S)$ the graph obtained by adding whiskers to all the vertices of S. After relabelling, we can always assume $S = \{x_1, \ldots, x_s\}$. The following lemma describes the connection between the whiskered graph $G \cup W(S)$ and Construction 3.1.

Lemma 4.2. *Let* G *be a finite simple graph on the vertex set* V *and let* $S \subseteq V$. *Let* χ *be the* s*-coloring of* $\mathrm{Ind}(G)|_S$ *given by* $S = \{x_1\} \cup \cdots \cup \{x_s\}$. *Then* $\mathrm{Ind}(G \cup W(S)) = \mathrm{Ind}(G)_\chi$.

Proof. The lemma is simply applying the definitions. □

Remark 4.3. One can also recover the clique-starring and clique-whiskering techniques of Woodroofe [16] and Cook and Nagel [6] from Construction 3.1, coloring all vertices and allowing each coloring class to have more than one vertex.

When restricted to independence complexes of graphs, Theorem 3.3 gives us necessary and sufficient conditions for a whiskered graph to be vertex decomposable. Below, $G \setminus \mu$ denotes a graph G with the vertices of μ and adjacent edges removed. For any subset $W \subseteq V$, we use $G|_W$ to denote the induced subgraph of G on W, i.e., the graph with vertices W and edge set $\{e \in E \mid e \subseteq W\}$.

Theorem 4.4. *Let* $\mathrm{Ind}(G)$ *be the independence complex of a graph* G *on a vertex set* V *and let* $S \subseteq V$. *Then* $\mathrm{Ind}(G \cup W(S))$ *is vertex decomposable if and only if* $\mathrm{Ind}((G \setminus \mu)|_{\overline{S}})$ *is vertex decomposable for all* $\mu \in \mathrm{Ind}(G)$ *with* $\mu \subseteq S$.

Proof. By Lemma 4.2, $\mathrm{Ind}(G \cup W(S)) = \mathrm{Ind}(G)_\chi$ for the s-coloring χ of $\mathrm{Ind}(G)_S$ given by $S = \{x_1\} \cup \cdots \cup \{x_s\}$. On the other hand, for any $\mu \subseteq S$, one can show that

$$\mathrm{link}_{\mathrm{Ind}(G)}(\mu)|_{\overline{S}} = \mathrm{Ind}(G \setminus \mu)|_{\overline{S}} = \mathrm{Ind}((G \setminus \mu)|_{\overline{S}}).$$

Thus the statement is simply restricting Theorem 3.3 to independence complexes. □

Recall that a graph G is *chordal* if every induced cycle of length ≥ 4 contains a chord. The independence complexes of chordal graphs are particularly nice:

Theorem 4.5 ([16, Corollary 7(2)]). *If G is chordal graph, then* $\mathrm{Ind}(G)$ *is vertex decomposable.*

We now show how Theorem 4.4 not only allows us to give a new proof of [8, Theorem 3.3], we can in fact strengthen it. The original conclusion of [8, Theorem 3.3] is that the associated edge ideals are sequentially Cohen-Macaulay. This now follows from Corollary 4.6 below and the fact that vertex decomposability implies sequentially Cohen-Macaulayness (see Remark 2.3).

Corollary 4.6. *Let G be a finite simple graph and let $S \subseteq V$. Suppose that $G \setminus S$, the induced subgraph over the vertices $V \setminus S$, is either a chordal graph or the five cycle C_5. Then* $\mathrm{Ind}(G \cup W(S))$ *is vertex decomposable.*

Proof. First assume that $G \setminus S$ is a chordal graph. Let $V = \{x_1, \ldots, x_n\}$ and suppose, after relabelling, $S = \{x_1, \ldots, x_s\}$. Let χ be the s-coloring of $\mathrm{Ind}(G)|_S$ given by $S = \{x_1\} \cup \cdots \cup \{x_s\}$. For any $\mu \subseteq S$, $(G \setminus \mu)|_{\overline{S}}$ is an induced subgraph of $G|_{\overline{S}}$, so it is chordal. By Theorem 4.5, $\mathrm{Ind}((G \setminus \mu)|_{\overline{S}})$ is vertex decomposable. Now apply Theorem 4.4.

The proof for the case $G \setminus S$ is a five-cycle is similar because the independence complex of a five-cycle is vertex decomposable, as are any induced subgraphs. □

Remark 4.7. We say that a graph $G = (V, E)$ is *bipartite* if there exists a partition $V = V_1 \cup V_2$ such that for all $e \in E$ we have $e \cap V_1 \neq \emptyset$ and $e \cap V_2 \neq \emptyset$. The fourth author [14] has shown that if G is bipartite, then the conditions of being vertex decomposable, shellable, and sequentially Cohen-Macaulay are all equivalent. Because all induced subgraphs of bipartite graphs are themselves bipartite, when we restrict Theorem 4.4 to bipartite graphs, we get a classification of which whiskered bipartite graphs are also shellable and sequentially Cohen-Macaulay, not just vertex decomposable.

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